J. Fluid Mech. (1967), *vol.* 28, *part* 1, *pp.* 131–147
Printed in Great Britain

Axially symmetric potential flow around a slender body

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(Received 15 June 1966)

Axially symmetric potential flow about an axially symmetric rigid body is considered. The potential due to the body is represented as a superposition of potentials of point sources distributed along a segment of the axis inside the body. The source strength distribution satisfies a linear integral equation. A complete uniform asymptotic expansion of its solution is obtained with respect to the slenderness ratio $\epsilon^{\frac{1}{2}}$, which is the maximum radius of the body divided by its length. The expansion contains integral powers of ϵ multiplied by powers of $\log \epsilon$. From it expansions of the potential, the virtual mass and the dipole moment of the body are obtained. The flow about the body in the presence of an axially symmetric stationary obstacle is also determined. The method of analysis involves a technique for the asymptotic solution of integral equations.

1. Introduction

Suppose an axially symmetric rigid body moves along its axis in an incompressible inviscid fluid at rest at infinity. The resulting irrotational motion of the fluid is an axially symmetric potential flow. We shall represent its potential Φ as a superposition of potentials of point sources distributed along a segment of the axis inside the body. Then the boundary condition on the surface of the body will lead to a linear integral equation for the source strength distribution. By using a special technique, we shall obtain a complete uniform asymptotic expansion of the solution of this integral equation with respect to the parameter ϵ, which is the square of the ratio of the maximum radius of the body to its length. From the solution we shall obtain asymptotic expansions of Φ, of the virtual mass M and of the dipole moment μ of the body. In addition we shall consider an arbitrary axially symmetric flow incident on the body and also the case when another stationary axially symmetric body is present.

Potential flow past an axially symmetric slender body has been studied extensively since the work of Munk (1924). This study is now part of slender body theory, which is the theory of any type of flow past a slender body. It is discussed in detail, together with the relevant references, in various books, e.g. Thwaites (1960). Apparently only the first two terms in the expansion with respect to the slenderness ratio $\epsilon^{\frac{1}{2}}$ had been determined before the work of Moran (1963), who obtained the third term for the axially symmetric case but was unable to go further. We shall obtain the complete expansion in this case. In doing so, like

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Moran, we shall use Landweber's (1951) idea that the source strength vanishes in an interval near each end of the body. The determination of the lengths of these intervals is part of the problem. We shall also assume that the profile curve of the body is analytic.

Our method for obtaining the asymptotic expansion of the solution of the integral equation may be of interest in itself because it seems to be applicable to other equations.

2. Derivation of the integral equation

Let $\Phi = \Phi^0 + \Phi^b$ be the potential of an irrotational axisymmetric flow of an incompressible, inviscid fluid past a slender body of revolution. The given function Φ^0 is the potential of the incident flow while Φ^b, the potential due to the presence of the body, is to be determined. As a consequence of our assumptions, both Φ^0 and Φ^b are harmonic functions in the exterior of the body with Φ^b vanishing at infinity. On the surface of the body, assumed to be fixed, the normal derivative of Φ must vanish.

It is convenient to introduce cylindrical co-ordinates (r, θ, x) with the origin at the body's nose and the x-axis along its symmetry axis. Then due to the axial symmetry of the flow, both Φ^0 and Φ^b are independent of θ and analytic in r^2 and x. If we use the length of the body as the unit of length, then the body intercepts the axis at $x = 0$ and $x = 1$. We write its profile curve as $r = \epsilon^{\frac{1}{2}}R(x)$ $(0 \leqslant x \leqslant 1)$, where $\epsilon^{\frac{1}{2}}$ is the slenderness ratio, i.e. the ratio of the maximum radius of the body to its length. Then $\max\limits_{x} R(x) = 1$. For a slender body, such as we shall consider, ϵ is small. We have introduced ϵ into the equation of the profile curve, so that $\Phi^b(x, r^2, \epsilon)$ will depend upon ϵ. Our objective is to obtain an asymptotic expansion of Φ^b with respect to ϵ around $\epsilon = 0$.

Let us define $S(x)$ by $S(x) = R^2(x)$. Thus $\pi\epsilon S(x)$ is the cross-sectional area of the body at x. We shall assume that $S(x)$ vanishes at $x = 0$ and $x = 1$ and is analytic in the interval $0 \leqslant x \leqslant 1$. Then it can be expanded in power series about the end-points as follows

$$S(x) = \sum_{n=1}^{\infty} c_n x^n, \quad S(x) = \sum_{n=1}^{\infty} d_n (1-x)^n. \tag{2.1}$$

Here $c_n = S^{(n)}(0)/n!$ and $d_n = (-1)^n S^{(n)}(1)/n!$. If the radii of curvature of the body are non-zero at the ends of the body then c_1 and d_1 are not zero.

We shall attempt to represent Φ^b as a superposition of point source potentials distributed along a segment of the x-axis inside the body with the unknown strength $f(x, \epsilon)$/unit length. Thus we write

$$\Phi(x, r^2, \epsilon) = \Phi^0(x, r^2) - \frac{1}{4\pi}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{f(\xi, \epsilon)\, d\xi}{[(x-\xi)^2 + r^2]^{\frac{1}{2}}}. \tag{2.2}$$

The constants $\alpha(\epsilon)$ and $\beta(\epsilon)$, which must be found in addition to $f(x, \epsilon)$, satisfy the inequalities $0 \leqslant \alpha \leqslant \beta \leqslant 1$. In terms of the Stokes stream function Ψ, related to Φ by $\Psi_x = -r\Phi_r$ and $\Psi_r = r\Phi_x$, we can rewrite (2.2) as

$$\Psi(x, r^2, \epsilon) = \Psi^0(x, r^2) - \frac{1}{4\pi}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{(x-\xi) f(\xi, \epsilon)\, d\xi}{[(x-\xi)^2 + r^2]^{\frac{1}{2}}}. \tag{2.3}$$

In obtaining (2.3) we have used the relation

$$\int_{\alpha}^{\beta} f(\xi,\epsilon)\,d\xi = 0. \tag{2.4}$$

This is a consequence of the fact that there is no flow of fluid through the surface of the body.

Since the axis is a streamline for both the incident and total flows, both Ψ^0 and Ψ are constant on that part of it outside the body. Therfore we may set $\Psi^0(x,0) = \Psi(x,0,\epsilon) = 0$. Furthermore, since the body is a continuation of the axial streamline for the total flow, we have $\Psi[x,\epsilon S(x),\epsilon] = 0$. Upon using this fact in (2.3), after setting $r^2 = \epsilon R^2(x) = \epsilon S(x)$, we obtain

$$\Psi^0[x,\epsilon S(x)] = \frac{1}{4\pi}\int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{(x-\xi)f(\xi,\epsilon)\,d\xi}{[(x-\xi)^2+\epsilon S(x)]^{\frac{1}{2}}}. \tag{2.5}$$

(2.5) is a linear integral equation from which we shall determine $f(x,\epsilon)$, $\alpha(\epsilon)$ and $\beta(\epsilon)$. Since the left side of (2.5) is analytic in x for $0 \leqslant x \leqslant 1$, $f(x,\epsilon)$ must be analytic in its domain of definition, $\alpha \leqslant x \leqslant \beta$. We assume that the coefficients in the expansion of $f(x,\epsilon)$ with respect to ϵ can be continued analytically throughout the interval $0 \leqslant x \leqslant 1$. We shall see that this assumption will enable us to determine α and β as power series in ϵ of the form

$$\alpha(\epsilon) = \sum_{n=1}^{\infty} \alpha_n \epsilon^n, \tag{2.6}$$

$$\beta(\epsilon) = 1 - \sum_{n=1}^{\infty} \beta_n \epsilon^n. \tag{2.7}$$

3. Asymptotic solution of the integral equation

To obtain an asymptotic expansion of the solution $f(x,\epsilon)$ of (2.5) with respect to ϵ around $\epsilon = 0$, we first expand each side of (2.5) with respect to ϵ, without taking account of the dependence of f on ϵ. The left side can be expanded as a power series in ϵ because Ψ^0 is analytic in r^2. The right side can be expanded asymptotically in powers of ϵ and powers of ϵ multiplied by $\log\epsilon$, as we shall show. The coefficients on the right side are linear expressions in f. Then (2.5) becomes

$$4\pi \sum_{j=1}^{\infty} \Psi_j(x)\,S^j(x)\,\epsilon^j \sim \int_{\alpha(\epsilon)}^{x} f\,d\xi - \int_{x}^{\beta(\epsilon)} f\,d\xi + \sum_{j=1}^{\infty} \epsilon^j(L_j + \log\epsilon\, G_j)f. \tag{3.1}$$

Here L_j and G_j are linear operators which we shall determine in § 4 and

$$\Psi_j(x) = \frac{1}{j!}\left(\frac{\partial}{\partial r^2}\right)^j \Psi^0(x,r^2)\big|_{r^2=0}. \tag{3.2}$$

We now differentiate (3.1) with respect to x to take account of the fact that the derivatives of the integrals are very simple

$$\frac{d}{dx}\left[\int_{\alpha(\epsilon)}^{x} f\,d\xi - \int_{x}^{\beta(\epsilon)} f\,d\xi\right] = 2f(x,\epsilon). \tag{3.3}$$

Then (3.1) becomes

$$4\pi \sum_{j=1}^{\infty} \frac{d}{dx}[\Psi_j(x) S^j(x)]\epsilon^j \sim 2f(x,\epsilon) + \sum_{j=1}^{\infty} \epsilon^j \frac{d}{dx}(L_j + \log\epsilon\, G_j)f. \tag{3.4}$$

To solve (3.4) we seek an asymptotic solution for f of the form

$$f(x,\epsilon) \sim \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} \epsilon^n (\log\epsilon)^m f_{nm}(x). \tag{3.5}$$

Here the $f_{nm}(x)$ are functions of x to be determined. Upon inserting (3.5) into (3.4) we obtain

$$4\pi \sum_{n=1}^{\infty} \frac{d}{dx}(\Psi_n S^n)\epsilon^n \sim 2\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\epsilon^n(\log\epsilon)^m f_{nm}(x) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{j=1}^{\infty}\epsilon^{j+n}(\log\epsilon)^m \times \frac{d}{dx}(L_j + \log\epsilon\, G_j)f_{nm}(x). \tag{3.6}$$

We now equate coefficients of $\epsilon^n(\log\epsilon)^m$ on the two sides of (3.6) and obtain the following system of equations:

$$f_{10} = 2\pi \frac{d}{dx}(\Psi_1 S), \tag{3.7}$$

$$f_{n0} = 2\pi \frac{d}{dx}(\Psi_n S^n) - \frac{1}{2}\sum_{j=1}^{n-1}\frac{d}{dx}(L_j f_{n-j,0}) \quad (n \geqslant 2), \tag{3.8}$$

$$f_{1m} = 0 \quad (m \geqslant 1), \tag{3.9}$$

$$f_{nm} = -\frac{1}{2}\left[\sum_{j=1}^{n-1}\frac{d}{dx}(L_j f_{n-j,m}) + \sum_{j=1}^{n-1}\frac{d}{dx}(G_j f_{n-j,m-1})\right] \quad (n \geqslant 2,\ m \geqslant 1). \tag{3.10}$$

We see from (3.7)–(3.10) that the f_{nm} can be determined recursively. Thus once the L_j and G_j are evaluated, (3.7)–(3.10) will yield the f_{nm} and (3.5) will be the asymptotic expansion of the solution.

Once $f(x,\epsilon)$ is found, we can use it to compute the dipole moment $\mu(\epsilon)$ of the source distribution, defined by

$$\mu(\epsilon) = -\frac{1}{4\pi}\int_{\alpha}^{\beta} x f(x,\epsilon)\,dx. \tag{3.11}$$

By using (3.5) in (3.11) and then expanding the resulting integrals in Taylor series with respect to ϵ, which occurs only in the limits of integration, we obtain

$$\mu(\epsilon) \sim -\frac{1}{4\pi}\sum_{s=1}^{\infty}\sum_{m=0}^{s-j-1}\sum_{j=0}^{s-1}\frac{\epsilon^s(\log\epsilon)^m}{j!}\frac{d^j}{d\epsilon^j}\left[\int_{\alpha(\epsilon)}^{\beta(\epsilon)} x f_{s-j,m}(x)\,dx\right]_{\epsilon=0}. \tag{3.12}$$

Similarly, by using the expansion (3.5) for f in (2.2), we obtain the following asymptotic expansion of Φ:

$$\Phi(x,r^2,\epsilon) \sim \Phi^0(x,r^2) - \frac{1}{4\pi}\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\epsilon^n(\log\epsilon)^m\int_{\alpha(\epsilon)}^{\beta(\epsilon)}\frac{f_{nm}(\xi)\,d\xi}{[(x-\xi)^2+r^2]^{\frac{1}{2}}}. \tag{3.13}$$

We can replace the integrals which appear in (3.13) by their Taylor series with respect to ϵ. The resulting asymptotic expansion of Φ, however, is no longer uniformly valid at the stagnation points. This non-uniform expansion is given by

$$\Phi(x,r^2,\epsilon) \sim \Phi^0(x,r^2) - \frac{1}{4\pi}\sum_{s=1}^{\infty}\sum_{m=0}^{s-j-1}\sum_{j=0}^{s-1}\frac{\epsilon^s(\log\epsilon)^m}{j!}\frac{d^j}{d\epsilon^j}\left[\int_{\alpha(\epsilon)}^{\beta(\epsilon)}\frac{f_{s-j,m}(\xi)\,d\xi}{[(x-\xi)^2+r^2]^{\frac{1}{2}}}\right]_{\epsilon=0}. \tag{3.14}$$

4. Asymptotic expansion of the integral operator

We shall now evaluate asymptotically in ϵ, around $\epsilon = 0$, the integral operator in the integral equation (2.5). Let us denote by $I(x,\epsilon)$ the integral operator applied to a function $F(x)$ which is independent of ϵ

$$I(x,\epsilon) = \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \frac{F(\xi)(x-\xi)\,d\xi}{[(x-\xi)^2+\epsilon S(x)]^{\frac{1}{2}}}. \tag{4.1}$$

We assume that $\alpha(\epsilon)$ and $\beta(\epsilon)$ are given by the power series (2.6) and (2.7) with given coefficients α_n and β_n. These coefficients will be determined later.

We begin by adding and subtracting integrals to (4.1) to obtain

$$I(x,\epsilon) = \int_{\alpha(\epsilon)}^{x} F(\xi)\,d\xi - \int_{x}^{\beta(\epsilon)} F(\xi)\,d\xi + \int_{\alpha}^{x} F(\xi)\left\{\frac{(x-\xi)}{[(x-\xi)^2+\epsilon S(x)]^{\frac{1}{2}}}-1\right\}d\xi$$
$$+\int_{x}^{\beta} F(\xi)\left\{\frac{(x-\xi)}{[(x-\xi)^2+\epsilon S(x)]^{\frac{1}{2}}}+1\right\}d\xi. \tag{4.2}$$

In the third integral on the right side of (4.2) we set $x-\xi = v$ and in the fourth integral we set $\xi - x = v$. We then obtain

$$I(x,\epsilon) = \int_{\alpha(\epsilon)}^{x} F(\xi)\,d\xi - \int_{x}^{\beta(\epsilon)} F(\xi)\,d\xi + W(x,\epsilon) + V(x,\epsilon). \tag{4.3}$$

Here

$$W(x,\epsilon) = \int_0^{x-\alpha} F(x-v)\,\{v(v^2+\epsilon S)^{-\frac{1}{2}}-1\}\,dv, \tag{4.4}$$

$$V(x,\epsilon) = -\int_0^{\beta-x} F(x+v)\,\{v(v^2+\epsilon S)^{-\frac{1}{2}}-1\}\,dv. \tag{4.5}$$

To find the asymptotic expansion of W and V, we are tempted to use in (4.4) and (4.5), the binomial expansion

$$v(v^2+\epsilon S)^{-\frac{1}{2}}-1 = \sum_{j=1}^{\infty}\left(\frac{\epsilon S}{v^2}\right)^j a_j, \quad a_j = (-1)^j(\tfrac{1}{2})(\tfrac{3}{2})\dots(\tfrac{1}{2}+j-1)/j!. \tag{4.6}$$

However, this expansion is not valid throughout the domains of integration since these domains include $v = 0$. Therefore we proceed in a different way. To $F(x+v)$ in the integrand of (4.5), we add and subtract the two leading terms in its Taylor series about $v = 0$. To $v[v^2+\epsilon S]^{-\frac{1}{2}}-1$, we add and subtract the leading term of its binomial expansion. In this manner we obtain

$$-V(x,\epsilon) = F(x)\int_0^{\beta-x}(v[v^2+\epsilon S]^{-\frac{1}{2}}-1)\,dv + F'(x)\int_0^{\beta-x} v(v[v^2+\epsilon S]^{-\frac{1}{2}}-1)\,dv$$
$$+a_1\epsilon S\int_0^{\beta-x}\left[F(x+v)-\sum_{j=0}^{1}F^{(j)}(x)\,v^j/j!\right]v^{-2}\,dv$$
$$+\int_0^{\beta-x}\left\{v[v^2+\epsilon S]^{-\frac{1}{2}}-\sum_{j=0}^{1}a_j\left(\frac{\epsilon S}{v^2}\right)^j\right\}\left[F(x+v)-\sum_{j=0}^{1}\frac{F^{(j)}(x)\,v^j/j!}{j!}\right]dv. \tag{4.7}$$

The last two integrals in (4.7) are finite because $F(x+v)-F(x)-F'(x)\,v = O(v^2)$ The first two integrals can be evaluated explicitly. The third contains ϵ only in $\beta(\epsilon)$ in the upper limit and can be expanded in powers of ϵ merely by successive differentiation and the use of the Taylor series. The fourth integral is the remainder. In (A 4) and (A 7) of appendix A it is shown that the first two integrals are $O(\epsilon^{\frac{1}{2}})$ and $O(\epsilon \log \epsilon)$ respectively. The third term is $O(\epsilon)$ since it contains ϵ as a factor. An analysis of the fourth integral by the method of appendix A shows that it is $O(\epsilon^{\frac{3}{2}})$, so that it is actually asymptotically negligible compared to the first three terms.

The foregoing procedure can now be applied to the fourth integral in (4.7). Thus to the first factor of the integrand we add and subtract the second term in the binomial expansion (4.6). To the second factor we add and subtract the next two terms of the Taylor expansion of $F(x+v)$ about $v = 0$. In this way we obtain a sum of integrals some of which can be evaluated explicitly, others of which can be expanded easily in power series in ϵ and a remainder which is asymptotically smaller than all these terms. The whole procedure can be re-applied to the new remainder, and this can be done repeatedly. In this way we can express $V(x,\epsilon)$ as a series of integrals which are of increasing order in ϵ, i.e. which are successively smaller for ϵ tending to zero. Exactly the same analysis can be applied to $W(x,\epsilon)$. In this way we obtain

$$\begin{aligned} W(x,\epsilon)+V(x\epsilon) &= \sum_{n=1}^{\infty} a_n S^n(x)\epsilon^n\{H_n(x,\epsilon)-\tilde{H}_n(x,\epsilon)\} \\ &+ \sum_{n=1}^{\infty}\frac{F^{(2n)}(x)}{(2n)!}\{P_n(x,\epsilon)-\tilde{P}_n(x,\epsilon)\}-\sum_{n=0}^{\infty}\frac{F^{(2n+1)}(x)}{(2n+1)!}\{K_n(x,\epsilon)+\tilde{K}_n(x,\epsilon)\}. \end{aligned} \tag{4.8}$$

Here H_n, P_n and K_n are defined by

$$H_n(x,\epsilon) = \int_0^{x-\alpha} v^{-2n}\left\{F(x-v)-\sum_{j=0}^{2n-1}\frac{F^{(j)}(x)v^j}{j!}\right\}dv, \tag{4.9}$$

$$P_n(x,\epsilon) = \int_0^{x-\alpha} v^{2n}\left\{v[v^2+\epsilon S]^{-\frac{1}{2}}-\sum_{j=0}^{n} a_j\left(\frac{\epsilon S}{v^2}\right)^j\right\}dv, \tag{4.10}$$

$$K_n(x,\epsilon) = \int_0^{x-\alpha} v^{2n+1}\left\{v[v^2+\epsilon S]^{-\frac{1}{2}}-\sum_{j=0}^{n} a_j\left(\frac{\epsilon S}{v^2}\right)^j\right\}dv \quad (a_0 = 1). \tag{4.11}$$

The functions $\tilde{H}_n$, $\tilde{P}_n$ and $\tilde{K}_n$ are defined by (4.9), (4.10) and (4.11) respectively with $x-\alpha$ replaced by $\beta - x$.

To obtain the asymptotic expansion of $W+V$ we must expand asymptotically the integrals (4.9)–(4.11). This is done in appendix B by evaluating (4.10) and (4.11) explicitly and then expanding the resulting functions, other than $\epsilon^{\frac{1}{2}}$ and $\log \epsilon$, in power series in ϵ. The integral (4.9) is expanded directly as a Taylor series in ϵ about $\epsilon = 0$. In these calculations it is helpful to use the results of appendix A, (A 4) and (A 7), which show that $P_n = O(\epsilon^{n+\frac{1}{2}})$, $K_n = O(\epsilon^{n+1}\log\epsilon)$ and the same for $\tilde{P}_n$ and $\tilde{K}_n$. Upon forming the difference $P_n - \tilde{P}_n$, which occurs in (4.8), we find that the fractional powers of ϵ cancel. Then, upon using the results of appendix B in (4.8), we obtain the following asymptotic expansion for $W+V$

$$W(x,\epsilon)+V(x,\epsilon) = \sum_{r=1}^{\infty}\epsilon^r[L_r+\log\epsilon G_r]F(x). \tag{4.12}$$

Here L_r and G_r are linear operators defined by the following equations:

$$
\begin{aligned}
L_1 F = \frac{1}{2}\Bigg\{ & F'(x)S(x)\left[\log\left(\frac{4x(1-x)}{S(x)}\right)-1\right]+F(x)\left[\frac{S(x)}{x}-\frac{S(x)}{1-x}\right] \\
& +S\left[\int_0^{1-x}\{F(x+v)-F(x)-F'(x)v\}v^{-2}\,dv\right. \\
& \left.-\int_0^{x}\{F(x-v)-F(x)+F'(x)v\}v^{-2}\,dv\right]\Bigg\},
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
L_r F = & \sum_{n=1}^{r-1}\sum_{i=0}^{r}\frac{F^{(2n)}(x)}{(2n)!\,(2n+1)}\{g_{r-i}t_{2n,i}-\tilde{g}_{r-i}\tilde{t}_{2n,i}\} \\
& +\sum_{n=1}^{r-1}\sum_{j=0}^{n}\frac{F^{(2n)}(x)}{(2n)!}\frac{a_j S^j}{2(n-j)+1}\{t_{2(n-j)+1,r-j}-\tilde{t}_{2(n-j)+1,r-j}\} \\
& +\sum_{n=1}^{r-1}\sum_{j=1}^{n}\sum_{i=0}^{r-j}\frac{(-S)^j F^{2n}(x)}{(2n+1)(2n)!}\prod_{m=1}^{j}\left(\frac{2n+1-2m}{2n+1-m}\right)\{g_{r-i-j}t_{2(n-j),i}-\tilde{g}_{r-i-j}\tilde{t}_{2(n-j),i}\} \\
& \qquad +F(x)[g_r-\tilde{g}_r+\alpha_r-\beta_r] \\
& -\sum_{n=0}^{r-2}\frac{(-S)^{n+1}F^{(2n+1)}(x)}{(2n+1)!}\prod_{m=0}^{n}\left(\frac{2n+1-2m}{2n+2-2m}\right)(h_{r-n-1}+\tilde{h}_{r-n-1}) \\
& -(-S)^r\prod_{m=0}^{r-1}\left(\frac{2r-1-2m}{2r-2m}\right)\frac{F^{(2r-1)}(x)}{(2r-1)!}\log\left\{\frac{4x(1-x)}{S(x)}\right\} \\
& -\sum_{n=1}^{r-1}\sum_{i=0}^{r}\frac{F^{(2n+1)}(x)}{(2n+1)!\,2(n+1)}[g_{r-i}t_{2n+1,i}+\tilde{g}_{r-i}\tilde{t}_{2n+1,i}] \\
& -\sum_{n=1}^{r-1}\sum_{j=0}^{n}\frac{a_j S^j F^{(2n+1)}(x)}{2(n-j+1)\,(2n+1)!}[t_{2(n-j+1),r-j}+\tilde{t}_{2(n-j+1),r-j}] \\
& -\frac{F'(x)}{2}\left[\sum_{i=0}^{r}(g_{r-i}t_{1,i}+\tilde{g}_{r-i}\tilde{t}_{1,i})-t_{2,r}-\tilde{t}_{2,r}\right] \\
& -\sum_{n=1}^{r-1}\sum_{j=1}^{n}\sum_{i=0}^{r-j}\frac{(-S)^j}{2(n+1)}\frac{F^{(2n+1)}(x)}{(2n+1)!}\prod_{m=1}^{j}\left(\frac{2n+3-2m}{2n+2-2m}\right) \\
& \qquad \times[g_{r-i-j}t_{2(n-j)+1,i}+\tilde{g}_{r-i-j}\tilde{t}_{2(n-j)+1,i}] \\
& -\sum_{n=1}^{r}\frac{a_n S^n}{(r-n)!}\frac{d^{r-n}}{d\epsilon^{r-n}}\left\{\int_0^{x-\alpha}\left[F(x-v)-\sum_{j=0}^{2n-1}\frac{F^{(j)}(x)-(v)^j}{j!}\right]v^{-2n}\,dv\right. \\
& \left.-\int_0^{\beta-x}\left[F(x+v)-\sum_{j=0}^{2n-1}\frac{F^{(j)}(x)\,v^j}{j!}\right]v^{-2n}\,dv\right\}_{\epsilon=0} \qquad (r \geqslant 2),
\end{aligned} \tag{4.14}
$$

$$
G_r F = (-S)^r\frac{F^{(2r-1)}(x)}{(2r-1)!}\prod_{m=0}^{r-1}\left(\frac{2r-1-2m}{2r-2m}\right) \qquad (r \geqslant 1). \tag{4.15}
$$

The functions g_k, $\tilde{g}_k$, t_{ij}, $\tilde{t}_{ij}$, h_k and $\tilde{h}_k$ in (4.14) are defined by (B 10)–(B 17). These functions have been eliminated from (4.13) by means of these definitions.

5. Determination of $\alpha(\epsilon)$ and $\beta(\epsilon)$

We now turn to the determination of $\alpha(\epsilon)$ and $\beta(\epsilon)$. Since $S(x)$ is analytic for $0 \leqslant x \leqslant 1$, (3.7) shows that $f_{10}(x)$ is also analytic in that interval. It then follows from the recursive nature of the system (3.7)–(3.10) that $f_{nm}(x)$ will be analytic for $0 \leqslant x \leqslant 1$ if $L_r F(x)$ and $G_r F(x)$ are analytic when $F(x)$ is, provided neither

c_1 nor d_1 is zero. We see from (4.13)–(4.15) that L_rF and G_rF involve integrals and derivatives of F multiplied by the functions $g_k(x)$, $\tilde{g}_k(x)$, $h_k(x)$, $\tilde{h}_k(x)$, $t_{ij}(x)$ and $\tilde{t}_{ij}(x)$. The t_{ij} and $\tilde{t}_{ij}$ are polynomials in x and are therefore analytic. We shall now examine the analyticity of g_k, $\tilde{g}_k$, h_k and $\tilde{h}_k$.

The functions $g_k(x)$ are defined in (B 9) as the coefficients in the expansion in powers of ϵ of the function $g(x,\epsilon) = [w(x,\epsilon)]^{\frac{1}{2}}$ where $w(x,\epsilon)$ is defined by

$$w(x,\epsilon) = [x-\alpha(\epsilon)]^2+\epsilon S(x). \tag{5.1}$$

Thus they depend upon the coefficients α_n in the expansion of $\alpha(\epsilon)$. They are singular at $x = 0$ except for certain values of α_n. To see this let us examine the first few $g_k(x)$, which are

$$g_0(x) = x, \quad g_1(x) = \tfrac{1}{2}S(x)x^{-1}\alpha_1, \quad g_2(x) = \tfrac{1}{2}[\alpha_1^2x^{-1}-2\alpha_2-(\tfrac{1}{2}S-\alpha_1x)^2x^{-3}],$$
$$g_3(x) = -\alpha_3+\alpha_1\alpha_2x^{-1}+\tfrac{1}{2}(\tfrac{1}{2}S-\alpha_1x)^3x^{-5}-\tfrac{1}{2}(\tfrac{1}{2}S-\alpha_1x)(\alpha_1^2-2\alpha_2x)x^{-3}. \tag{5.2}$$

The function $g_0(x)$ is obviously regular and so is $g_1(x)$ because $S(0) = 0$. By using the expansion (2.1) of $S(x)$ about $x = 0$, we can separate $g_2(x)$ and $g_3(x)$ into singular and regular parts, the latter denoted by $\gamma_2(x)$ and $\gamma_3(x)$:

$$g_2(x) = x^{-1}\{\alpha_1^2-(\tfrac{1}{2}c_1-\alpha_1)^2\}+\gamma_2(x), \tag{5.3}$$

$$g_3(x) = \tfrac{1}{2}x^{-2}\{(\tfrac{1}{2}c_1-\alpha_1)^3-(\tfrac{1}{2}c_1-\alpha_1)\alpha_1^2\}$$
$$+\tfrac{1}{2}x^{-1}\{\tfrac{1}{2}\alpha_2c_1+\tfrac{1}{2}c_2\alpha_1^2+\tfrac{3}{16}c_2c_1^2-\tfrac{3}{4}c_1c_2\alpha_1\}+\gamma_3(x). \tag{5.4}$$

The singular term in (5.3) vanishes if we choose for α_1 the value

$$\alpha_1 = \tfrac{1}{4}c_1. \tag{5.5}$$

When α_1 has this value the x^{-2} term in (5.4) vanishes. The x^{-1} in (5.4) can be eliminated by setting

$$\alpha_2 = -\tfrac{1}{16}c_1c_2. \tag{5.6}$$

By making the functions $g_k(x)$ regular at $x = 0$ we can determine the further terms in the expansion of $\alpha(\epsilon)$. In this way we find

$$\alpha(\epsilon) = \tfrac{1}{4}c_1\epsilon-\tfrac{1}{16}c_1c_2\epsilon^2+\tfrac{1}{64}\epsilon^3(c_1^2c_3+2c_1c_2^2)$$
$$-\tfrac{1}{256}\epsilon^4(c_1^3c_4+7c_1^2c_2c_3+5c_1c_2^3)+O(\epsilon^5). \tag{5.7}$$

In a similar way, by making the $\tilde{g}_k(x)$ regular at $x = 1$, we obtain

$$\beta(\epsilon) = 1-\tfrac{1}{4}d_1\epsilon+\tfrac{1}{16}d_1d_2\epsilon^2-\tfrac{1}{64}\epsilon^3(d_1^2d_3+2d_1d_2^2)$$
$$+\tfrac{1}{256}\epsilon^4(d_1^3d_4+7d_1^2d_2d_3+5d_1d_2^3)+O(\epsilon^5). \tag{5.8}$$

The results (5.7) and (5.8) agree with those of Moran (1963), obtained in a somewhat more complicated way.

To prove that all the $g_k(x)$ can be made regular at $x = 0$ by appropriate choice of α_k, it is convenient to express the g_k in terms of $g^{(k)}(x,\epsilon) = \partial^k g(x,\epsilon)/\partial\epsilon^k$ by the relation $g_k(x) = g^{(k)}(x,0)/k!$. Then $g_k(x)$ will be regular at $x = 0$ if $g^{(k)}(x,0)$ is. Now we can readily compute the first few $g^{(k)}(x,\epsilon)$ and observe that they are given by the following recursive formula, which we can then prove by induction,

$$g^{(k)} = w^{(k)}/2w^{\frac{1}{2}}+w^{-\frac{1}{2}}\sum_{j=1}^{k-1} a_{kj}g^{(k-j)}g^{(j)} \quad (k \geqslant 1). \tag{5.9}$$

Here the a_{kj} are some constants and $w^{(k)} = \partial^k w/\partial\epsilon^k$. From the definition of w we have

$$w^{(k)}(x,0) = -2k!\,\alpha_k x + k!\sum_{j=1}^{k-1}\alpha_{k-j}\alpha_j + \delta_{k1}S(x) \quad (k \geqslant 1). \tag{5.10}$$

In addition $w^{\frac{1}{2}}(x,0) = x$. By using this together with (5.10) in (5.9), and setting $\epsilon = 0$, we obtain

$$g^{(k)}(x,0) = -k!\,\alpha_k + \frac{1}{x}\sum_{j=1}^{k-1}\left[\frac{k!}{2}\alpha_{k-j}\alpha_j + a_{kj}g^{(k-j)}(x,0)\,g^{(j)}(x,0)\right] + \frac{\delta_{k1}S(x)}{2x} \quad (k \geqslant 1). \tag{5.11}$$

For $k = 1$ the sum in (5.11) is vacuous and $g^{(1)}(x,0)$ is regular at $x = 0$. For $k > 1$, $g^{(k)}(x,0)$ will be regular if the sum in (5.11) vanishes at $x = 0$. Setting the sum equal to zero at $x = 0$ yields an equation which we can solve for α_{k-1} provided $\alpha_1 \neq 0$, with the result

$$\alpha_{k-1} = -\frac{1}{\alpha_1}\left[\sum_{j=2}^{k-2}\frac{k!}{2}\alpha_{k-j}\alpha_j + \sum_{j=1}^{k-1}a_{kj}g^{(k-j)}(0,0)\,g^{(j)}(0,0)\right] \quad (k \geqslant 2). \tag{5.12}$$

Thus if $\alpha_1 \neq 0$, and if the $g^{(j)}(x,0)$ with $j = 1, \ldots, k-1$ are regular at $x = 0$, it follows that $g^{(k)}(x,0)$ is also regular there provided α_k is given by the finite expression (5.12). Since $g^{(1)}(x,0)$ is regular, we conclude by induction that all the $g^{(k)}(x,0)$ can be made regular at $x = 0$ by using (5.12) to determine the α_k recursively. A similar analysis shows that the $\tilde{g}_k(x)$ can be made regular at $x = 1$ by choosing the β_k appropriately.

If $\alpha_1 = 0$, which is the case if $c_1 = 0$, then all the $g_k(x)$ are regular at $x = 0$ if $\alpha(\epsilon) \equiv 0$. To show this we note that when $c_1 = 0$ and $\alpha \equiv 0$, w is given by

$$w(x,\epsilon) = x^2 + \epsilon S(x) = x^2 + \epsilon\sum_{n=2}^{\infty}c_n x^n. \tag{5.13}$$

Thus $g(x,\epsilon)$ is given by

$$g(x,\epsilon) = w^{\frac{1}{2}} = x\left[1 + \epsilon\sum_{n=2}^{\infty}c_n x^{n-2}\right]^{\frac{1}{2}}. \tag{5.14}$$

It follows from (5.14) that all the $g_k(x)$ are regular at $x = 0$, which proves that $\alpha(\epsilon) \equiv 0$. If $d_1 = 0$ it follows in the same way that all the $\tilde{g}_k(x)$ are regular at $x = 1$ provided $\beta(\epsilon) \equiv 0$. When $c_1 = 0$, the $f_{nm}(x)$ are not analytic at $x = 0$ and when $d_1 = 0$, they are not analytic at $x = 1$.

The functions $h_k(x)$ are defined in terms of the square root in (5.1). Once the α_n are chosen to make the $g_k(x)$ regular, it follows that the $h_k(x)$ are also regular. Similarly, the $\tilde{h}_k(x)$ are regular if the $\tilde{g}_k(x)$ are. Therefore, once α_n and β_n are determined to make $g_k(x)$ and $\tilde{g}_k(x)$ regular, it follows that $L_r F(x)$ and $G_r F(x)$ are analytic whenever $F(x)$ is. Thus, the $f_{nm}(x)$ are analytic for $0 \leqslant x \leqslant 1$.

6. Body in a uniform stream

Let us now apply our results to the case of a body moving with speed $-U$ along its axis in a fluid at rest at infinity. In a co-ordinate system in which the body is at rest, the fluid has the velocity U at infinity parallel to the axis and the incident stream function is $\Psi^0(x,r^2) = \frac{1}{2}Ur^2$. The coefficients $\Psi_j(x)$, which are the coefficients in the expansion of Ψ^0 in powers of r^2, are

$$\Psi_1(x) = \tfrac{1}{2}U, \quad \Psi_j(x) = 0 \quad (j \geqslant 2). \tag{6.1}$$

We insert these coefficients into (3.7)–(3.10) together with the expressions (4.13)–(4.15) defining L_j and G_j. Then we use these equations to calculate the following five $f_{nm}(x)$:

$$f_{10}(x) = \pi U S'(x), \tag{6.2}$$

$$f_{21}(x) = -\frac{1}{2}\frac{d}{dx}G_1 f_{10} = \frac{\pi}{4}U\frac{d}{dx}\left\{S\frac{d^2S}{dx^2}\right\}, \tag{6.3}$$

$$\begin{aligned} f_{20}(x) = -\frac{1}{2}\frac{d}{dx}L_1 f_{10} = -\frac{\pi}{4}U\frac{d}{dx}\Bigg\{&\frac{SS'}{x}-\frac{SS'}{1-x}-SS''+S''S\log\left[\frac{4x(1-x)}{S(x)}\right] \\ &+S\int_0^{1-x}[S'(x+v)-S'(x)-S''(x)v]\,v^{-2}\,dv \\ &-S\int_0^{x}[S'(x-v)-S'(x)+S''(x)v]\,v^{-2}\,dv\Bigg\}, \end{aligned} \tag{6.4}$$

$$f_{32} = -\frac{1}{2}\frac{d}{dx}G_1 f_{21} = \frac{\pi}{16}\frac{d}{dx}\left\{S\frac{d^2}{dx^2}\left(S\frac{d^2S}{dx^2}\right)\right\}, \tag{6.5}$$

$$\begin{aligned} f_{31} = -\frac{1}{2}\frac{d}{dx}\{L_1 f_{21}+G_2 f_{10}+G f_{20}\} = \frac{1}{4}\Bigg\{&\frac{d}{dx}\left(S\frac{df_{20}}{dx}\right)-\frac{1}{8}\frac{d}{dx}\left(S^2\frac{d^3f_{10}}{dx^3}\right) \\ &-\frac{d}{dx}\left[\left\{S\log\left(\frac{4x(1-x)}{S}\right)-S\right\}\frac{df_{21}}{dx}+\left(\frac{S}{x}-\frac{S}{1-x}\right)f_{21}\right. \\ &+S\int_0^{x}[f_{21}(x-v)-f_{21}(x)+f_{21}'(x)v]\,v^{-2}\,dv \\ &-S\int_0^{1-x}[f_{12}(x+v)-f_{21}-f_{21}'v]\,v^{-2}\,dv\Bigg\}. \end{aligned} \tag{6.6}$$

To make the expression for f_{31} more explicit, we may insert the preceding expressions into it. When (6.2)–(6.6) are used in (3.5) they yield the expansion for the source distribution $f(x,\epsilon)$ up to $O(\epsilon^3)$. The first three of these terms were obtained by Moran (1963). His expression for f_{20} involves an infinite series, rather than the integrals in (6.4), to which the series is equal.

As an application of the results (6.2)–(6.6) we can compute the dipole moment μ of the source distribution, given by (2.12). From the dipole moment we can find the added or virtual mass M of the body by using Taylor's (1928) formula

$$M = 4\pi\rho\mu U^{-1}-\rho V. \tag{6.7}$$

Here ρ is the density of the fluid and V is the volume of the body. By using (6.2)–(6.6), (3.12) and (6.7) we obtain

$$\begin{aligned} \rho^{-1}M = -\tfrac{1}{4}\pi\epsilon^2\log\epsilon\int_0^1[S'(x)]^2dx+\frac{\pi\epsilon^2}{4}\Bigg(&\int_0^1\Bigg\{\frac{SS'}{1-x}-\frac{SS'}{x}+SS''-SS''\log\left[\frac{4x(1-x)}{S}\right] \\ &+S\int_0^x[S'(x-v)-S'(x)+S''(x)v]\,v^{-2}\,dv \\ &-S\int_0^{1-x}[S'(x+v)-S'(x)-S''(x)v]\,v^{-2}\,dv\Bigg\}dx-d_1^2\Bigg) \\ &-\frac{\pi}{16}\epsilon^3(\log\epsilon)^2\int_0^1 S'\frac{d^2}{dx^2}(SS'')\,dx+O(\epsilon^3\log\epsilon). \end{aligned} \tag{6.8}$$

7. Slender body approaching an obstacle

We now consider the flow which results when a slender body moves along its axis toward or away from a stationary obstacle which is symmetric about the same axis. Let D denote the domain exterior to the obstacle and let Σ denote the obstacle surface. We introduce the same cylindrical co-ordinate system as before and we denote the velocity of the body along the axis by $-U$. To express the potential of the flow by an axial source distribution, we shall utilize the Neumann function $N(x, r^2, \xi)$ for the domain D. This is a function harmonic in D, except at the point $x = \xi$, $r = 0$ where it has the singularity of a point source; it vanishes at infinity and has a vanishing normal derivative on Σ. It can be written as

$$N(x, r^2, \xi) = \tfrac{1}{4}\pi^{-1}[(x-\xi)^2+r^2]^{-\frac{1}{2}}+\tfrac{1}{4}\pi^{-1}\tilde{N}(x, r^2, \xi). \tag{7.1}$$

Here $\tilde{N}$ is the regular harmonic part of N, which depends upon the distance between the moving body and the fixed object because the origin of co-ordinates is at the nose of the body.

In terms of N we write Φ in the form

$$\Phi(x, r^2, \epsilon) = \frac{1}{4\pi}\int_\alpha^\beta f(\xi, \epsilon)\,[(x-\xi)^2+r^2]^{-\frac{1}{2}}\,d\xi+\frac{1}{4\pi}\int_\alpha^\beta f(\xi, \epsilon)\,\tilde{N}(x, r^2, \xi)\,d\xi. \tag{7.2}$$

This expression is harmonic outside the body; has vanishing normal derivative on Σ and vanishes at infinity. Upon requiring that its normal derivative on the body equal the normal velocity of the body we obtain

$$\begin{aligned}2\pi U\epsilon S'(x) = {}&\int_\alpha^\beta \frac{f(\xi, \epsilon)\{\epsilon S(x)-\frac{1}{2}(x-\xi)\,\epsilon S'(x)\}\,d\xi}{[(x-\xi)^2+\epsilon S(x)]^{\frac{3}{2}}}\\&+\epsilon\int_\alpha^\beta \left\{\frac{S'(x)}{2}\frac{\partial\tilde{N}}{\partial x}-2S(x)\frac{\partial\tilde{N}}{\partial r^2}\right\}_{r^2=\epsilon S(x)} f(\xi, \epsilon)\,d\xi.\end{aligned} \tag{7.3}$$

This is a linear integral equation for the determination of $f(x, \epsilon)$.

We now integrate (7.3) with respect to x from 0 to x and use (2.4) to obtain

$$2\pi U\epsilon S(x) = \int_\alpha^\beta \frac{f(\xi, \epsilon)\,(x-\xi)\,d\xi}{[(x-\xi)^2+\epsilon S(x)]^{\frac{1}{2}}}+\epsilon\int_\alpha^\beta f(\xi, \epsilon)\,\Gamma(x, \xi, \epsilon)\,d\xi. \tag{7.4}$$

Here we have introduced $\Gamma(x, \xi, \epsilon)$ which is defined by

$$\Gamma(x, \xi, \epsilon) = \epsilon\int_0^x \left\{\frac{S'(t)}{2}\frac{\partial\tilde{N}(t, r^2, \xi)}{\partial t}-2S(t)\frac{\partial\tilde{N}(t, r^2, \xi)}{\partial r^2}\right\}_{r^2=\epsilon S(t)} dt. \tag{7.5}$$

Since $\tilde{N}$ is a harmonic function it follows that Γ is a regular function of ϵ which we can expand in a Taylor series

$$\Gamma(x, \xi, \epsilon) = \sum_{j=1}^{\infty} \Gamma_j(x, \xi)\,\epsilon^j, \quad \Gamma_j = \frac{1}{j!}\frac{\partial^j}{\partial\epsilon^j}\Gamma(x, \xi, \epsilon)|_{\epsilon=0}. \tag{7.6}$$

The functions $\Gamma_j(x, \xi)$ are analytic functions of x for $0 \leqslant x \leqslant 1$ and $\alpha \leqslant \xi \leqslant \beta$. Upon using (7.6) in (7.4) we obtain

$$2\pi U\epsilon S = \int_\alpha^\beta \frac{f(\xi, \epsilon)\,(x-\xi)\,d\xi}{[(x-\xi)^2+\epsilon S(x)]^{\frac{1}{2}}}+\sum_{j=1}^{\infty}\epsilon^j\int_\alpha^\beta f(\xi, \epsilon)\,\Gamma_j(x, \xi)\,d\xi. \tag{7.7}$$

The integral operators in the sum in (7.7) can be expanded in Taylor series with respect to ϵ, which occurs only in the limits of integration. Then this operator, when applied to a function $F(x)$ independent of ϵ, can be written as

$$\sum_{j=1}^{\infty} \epsilon^j \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \Gamma_j(x,\xi)\, F(\xi)\, d\xi = \sum_{r=1}^{\infty} \epsilon^r Q_r F. \tag{7.8}$$

The operators Q_r, which are independent of ϵ, are defined by

$$Q_r F = \sum_{p=0}^{r-1} \frac{1}{p!} \left[\frac{\partial^p}{\partial \epsilon^p} \int_{\alpha(\epsilon)}^{\beta(\epsilon)} \Gamma_{r-p}(x,\xi)\, F(\xi)\, d\xi \right]_{\epsilon=0}. \tag{7.9}$$

The first integral in (7.7) has been analysed already and can be written as shown on the right side of (3.1). Upon using that result and (7.8) in (7.7), and then differentiating with respect to x, we obtain

$$2\pi U \epsilon S'(x) \sim 2f(x,\epsilon) + \sum_{j=1}^{\infty} \epsilon^j \frac{d}{dx} (L_j + Q_j + G_j \log \epsilon) f. \tag{7.10}$$

This equation is the same as (3.4) with L_j replaced by $L_j + Q_j$, $\Psi_1 = \frac{1}{2}U$ and $\Psi_j = 0$ for $j > 1$. Therefore we seek an asymptotic solution for $f(x,\epsilon)$ of the same form (3.5). We again obtain for the coefficients $f_{nm}(x)$ the equations (3.7)–(3.10) with these same replacements.

It follows from these equations that f_{10}, f_{21} and f_{32} are given by (6.2), (6.3) and (6.5) and are therefore unaffected by the obstacle. All the other f_{nm} will be modified. For example, f_{20} is given by the right side of (6.4) plus an additional term which we will call f_{20}^{+}. It is

$$f_{20}^{+} = -\frac{\pi U}{2} \frac{d}{dx} \int_0^1 \Gamma_1(x,v)\, S'(v)\, dv. \tag{7.11}$$

Once the f_{nm} have been found, (7.2) yields Φ and (3.12) yields the asymptotic expansion of the dipole moment of the body. The asymptotic expansion of Φ is of the form (3.13) with $\frac{1}{4}\pi^{-1}[(x-\xi)^2 + r^2]^{-\frac{1}{2}}$ replaced by $N(x, r^2, \xi)$.

To find the force F on the body, which is along the axis, we use the Bernoulli equation for the pressure to obtain

$$F = \pi \epsilon \rho \int_0^1 [\Phi_t + \tfrac{1}{2}(\nabla\Phi)^2]\, S'(x)\, dx. \tag{7.12}$$

Since Φ is proportional to the velocity U, Φ_t contains a term proportional to the acceleration U_t. The coefficient of U_t in the expression for F is the added mass M given by

$$\begin{aligned} M &= \frac{\epsilon\rho}{4U} \int_0^1 \Phi[x, \epsilon S(x), \epsilon]\, S'(x)\, dx \\ &= \frac{\epsilon\rho}{4U} \int_0^1 S'(x) \int_\alpha^\beta f(\xi,\epsilon) \{[(x-\xi)^2 + \epsilon S(x)]^{-\frac{1}{2}} + \tilde{N}(x, \epsilon S, \xi)\}\, d\xi\, dx. \end{aligned} \tag{7.13}$$

In obtaining (7.13) we have used (7.2) and taken account of the rate of change of the distance between the moving body and the fixed object.

To evaluate the x integral of the first term in the second integrand, we add and subtract $2(x-\xi)$ and integrate to get

$$\frac{1}{4}\int_0^1 \epsilon S'(x)\,[(x-\xi)^2+\epsilon S]^{-\frac{1}{2}}\,dx = \tfrac{1}{2}-\xi-\frac{1}{2}\int_0^1 (x-\xi)\,[(x-\xi)^2+\epsilon S]^{-\frac{1}{2}}\,dx. \tag{7.14}$$

Now we multiply (7.14) by $f(\xi)$ and integrate with respect to ξ, obtaining

$$\frac{1}{4}\int_\alpha^\beta\int_0^1 f(\xi)\,\epsilon S'(x)\,[(x-\xi)^2+\epsilon S]^{-\frac{1}{2}}\,dx\,d\xi = \frac{1}{2}\int_\alpha^\beta f(\xi)\,d\xi-\int_\alpha^\beta \xi f(\xi)\,d\xi$$
$$-\frac{1}{2}\int_\alpha^\beta\int_0^1 f(\xi)\,(x-\xi)\,[(x-\xi)^2+\epsilon S]^{-\frac{1}{2}}\,dx\,d\xi. \tag{7.15}$$

The first integral on the right vanishes, the second is $4\pi\mu$ and the last can be found by integrating the integral equation (7.4) from 0 to 1 with respect to x. Then using (7.5), (7.15) becomes

$$\frac{1}{4}\int_\alpha^\beta\int_0^1 f(\xi)\,\epsilon S'(x)\,[(x-\xi)^2+\epsilon S]^{-\frac{1}{2}}\,dx\,d\xi = 4\pi\mu - VU$$
$$+\frac{\epsilon}{2}\int_0^1\int_\alpha^\beta f(\xi)\,(1-x)\left\{\frac{S'}{2}\frac{\partial\tilde{N}}{\partial x}-2S\frac{\partial\tilde{N}}{\partial r^2}\right\}_{r^2=\epsilon S(x)} d\xi\,dx. \tag{7.16}$$

Upon using (7.16) in (7.13) we obtain

$$M = 4\pi\rho U^{-1}-\rho V+\frac{\epsilon\rho}{4U}\int_\alpha^\beta f(\xi)\int_0^1\left\{\tilde{N}S'+(1-x)\left[S'\frac{\partial\tilde{N}}{\partial x}-4S\frac{\partial\tilde{N}}{\partial r^2}\right]\right\}_{r^2=\epsilon S(x)} dx\,d\xi. \tag{7.17}$$

We shall now assume that $\tilde{N}$ can be represented as a superposition of potentials due to point sources distributed along the axis of the obstacle, and inside it, with density $p(\xi,\eta)$. Thus

$$\tilde{N}(x,r^2,\xi) = \frac{1}{4\pi}\int_\gamma^\delta p(\xi,\eta)\,[(x-\eta)^2+r^2]^{-\frac{1}{2}}\,d\eta. \tag{7.18}$$

Then (7.17) becomes

$$M = 4\pi\rho\mu U^{-1}-\rho V+\frac{\rho}{16\pi U}\int_\alpha^\beta f(\xi)\int_\gamma^\delta p(\xi,\eta)\int_0^1 \{\epsilon S'[(x-\eta)^2+\epsilon S]^{-\frac{1}{2}}$$
$$+(1-x)\,(-\epsilon S'[x-\eta]+2\epsilon S)\,[(x-\eta)^2+\epsilon S]^{-\frac{3}{2}}\}\,dx\,d\eta\,d\xi. \tag{7.19}$$

We now add and subtract $2(1-x)\,(x-\eta)^2\,[(x-\eta)^2+\epsilon S]^{-\frac{3}{2}}$ to the last term in the integrand in (7.19). Then we can write the x integral in (7.19) as follows, and integrate the last term by parts, cancel terms and integrate again. In this way we obtain upon taking account of the fact that $\eta<0$,

$$\begin{aligned}
&\int_0^1 \{\epsilon S'[(x-\eta)^2+\epsilon S]^{-\frac{1}{2}}+2(1-x)\,[\epsilon S+(x-\eta)^2]\,[(x-\eta)^2+\epsilon S]^{-\frac{3}{2}}\\
&\qquad\qquad +(1-x)\,(x-\eta)\,(-\epsilon S'-2x+2\eta)\,[(x-\eta)^2+\epsilon S]^{-\frac{3}{2}}\}\,dx\\
&= \int_0^1 [\epsilon S'+2(1-x)]\,[(x-\eta)^2+\epsilon S]^{-\frac{1}{2}}\,dx+[2(1-x)\,(x-\eta)\,[(x-\eta)^2+\epsilon S]^{-\frac{1}{2}}]_0^1\\
&\qquad\qquad -\int_0^1 [2(1-x)-2(x-\eta)]\,[(x-\eta)^2+\epsilon S]^{-\frac{1}{2}}\,dx,\\
&= \int_0^1 [\epsilon S'+2(x-\eta)]\,[(x-\eta)^2+\epsilon S]^{-\frac{1}{2}}\,dx-2,\\
&= 2[(x-\eta)^2+\epsilon S]^{\frac{1}{2}}\big|_0^1-2 = 2(1-\eta)+2\eta-2,\\
&= 0.
\end{aligned} \tag{7.20}$$

Now (7.19) becomes

$$M = 4\pi\rho\mu U^{-1} - \rho V. \tag{7.21}$$

This is the same as Taylor's formula (6.7) with μ the dipole moment of the source distribution inside the body, which is no longer equal to the dipole strength of the flow far from the body and obstacle.*

As an example of the above results, let us suppose the obstacle is the plane $x = -a$. Then the method of images yields

$$\tilde{N}(x, r^2, \xi) = [(x+\xi+2a)^2 + r^2]^{-\frac{1}{2}}. \tag{7.22}$$

From (7.5) and (7.22) we find

$$\Gamma(x, \xi, \epsilon) = \frac{(x+\xi+2a)}{[(x+\xi+2a)^2 + \epsilon^2 S]^{\frac{1}{2}}} - 1. \tag{7.23}$$

For $a > 0$ and ϵ small, we expand (7.23) by the binomial theorem and find that (7.6) holds with

$$\Gamma_j = a_j \frac{S^j(x)}{(x+\xi+2a)^{2j}} \quad (j = 1, 2, \ldots). \tag{7.24}$$

The binomial coefficients a_j are given by (4.6). By using Γ_1 in (7.11) we obtain

$$f_{20}^{+} = \frac{\pi U}{4} \frac{d}{dx}\left\{S(x)\int_0^1 \frac{S'(v)\,dv}{(x+v+2a)}\right\}. \tag{7.25}$$

The corresponding addition to the dipole moment, which we denote by μ_{20}^{+}, is

$$\mu_{20}^{+} = -\frac{1}{4\pi}\int_0^1 x f_{20}^{+}(x)\,dx = \frac{U}{16}\int_0^1 S(x)\int_0^1 \frac{S'(v)}{(x+v+2a)}\,dv\,dx. \tag{7.26}$$

From (6.7) we see that this term in the dipole moment contributes a term $M_{20}^{+} = 4\pi\mu_{20}^{+}U^{-1}$ to the virtual mass. If the plane obstacle represents a free surface on which $\Phi = 0$, the results (7.22)–(7.26) all hold provided the right sides are multiplied by -1.

The research in this paper was supported by the National Science Foundation under grant no. GP 3668.

Appendix A. Asymptotic estimates of P_n and K_n

We shall now estimate the functions $P_n(x, \epsilon)$ and $K_n(x, \epsilon)$, defined by (4.10) and (4.11), for small values of ϵ. In (4.10) we introduce the integration variable $u = (\epsilon S)^{-\frac{1}{2}} v$ and (4.10) becomes

$$P_n(x, \epsilon) = (\epsilon S)^{n+\frac{1}{2}} \int_0^{(x-\alpha)(\epsilon S)^{-\frac{1}{2}}} u^{2n}\left[u(1+u^2)^{-\frac{1}{2}} - \sum_{j=0}^{n} a_j u^{-2j}\right] du. \tag{A 1}$$

The integrand is of one sign for all values of $u > 0$ and is $O(u^{-2})$ as u becomes infinite. Therefore the right side of (A 1) is increased in magnitude if the upper limit is replaced by infinity and the resulting integral converges. Thus we conclude from (A 1) that for all $x \geqslant \alpha$ and ϵ small,

$$P_n(x, \epsilon) = O(\epsilon^{n+\frac{1}{2}}). \tag{A 2}$$

* [*Note added in proof.*] Equation (7.21) is a special case of the generalized Taylor formula given by Landweber & Yih (1956).

The same change of variable in (4.11) yields

$$K_n(x,\epsilon) = (\epsilon S)^{n+1}\int_0^{(x-\alpha)(\epsilon S)^{-\frac{1}{2}}} u^{2n+1}\left[u(1+u^2)^{-\frac{1}{2}} - \sum_{j=0}^{n} a_j u^{-2j}\right] du. \quad \text{(A 3)}$$

The integrand in (A 3) is also of one sign and behaves like $a_{n+1}u^{-1}$ as u becomes infinite. Thus by splitting the integral at $u = 1$ and adding and subtracting $a_{n+1}u^{-1}$ to the second integral we can write (A 3) as

$$\begin{aligned} K_n(x,\epsilon) = {} & (\epsilon S)^{n+1}\int_0^1 u^{2n+1}\left[u(1+u^2)^{-\frac{1}{2}} - \sum_{j=0}^{n} a_j u^{-2j}\right] du \\ & + (\epsilon S)^{n+1}\int_1^{(x-\alpha)(\epsilon S)^{-\frac{1}{2}}} u^{2n+1}\left[u(1+u^2)^{-\frac{1}{2}} - \sum_{j=0}^{n+1} a_j u^{-2j}\right] du \\ & + (\epsilon S)^{n+1}\int_1^{(x-\alpha)(\epsilon S)^{-\frac{1}{2}}} a_{n+1}u^{-1}\, du. \end{aligned} \quad \text{(A 4)}$$

The first integral is dependent of ϵ, the second is less in magnitude than the corresponding finite integral with upper limit infinity and the last integral is $O(\log\epsilon)$. Thus (A 4) yields

$$K_n(x,\epsilon) = O(\epsilon^{n+1}\log\epsilon). \quad \text{(A 5)}$$

The result (A 2) also holds for $\tilde{P}_n$ and (A 5) holds for $\tilde{K}_n$, as we see from their definitions.

Appendix B. Asymptotic expansion of K_n, P_n and H_n

To expand the integrals K_n and P_n asymptotically for ϵ small we first evaluate them by using the following relations

$$\int_0^{x-\alpha} \frac{dv}{(v^2+\epsilon S)^{\frac{1}{2}}} = \log\{(x-\alpha) + ((x-\alpha)^2+\epsilon S)^{\frac{1}{2}}\} - \tfrac{1}{2}\log(\epsilon S), \quad \text{(B 1)}$$

$$\int_0^{x-\alpha} \frac{v\,dv}{(v^2+\epsilon S)^{\frac{1}{2}}} = [(x-\alpha)^2+\epsilon S]^{\frac{1}{2}} - (\epsilon S)^{\frac{1}{2}}, \quad \text{(B 2)}$$

$$\int_0^{x-\alpha} \frac{v^m\,dv}{(v^2+\epsilon S)^{\frac{1}{2}}} = \frac{[(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}}{m}(x-\alpha)^{m-1} - \frac{(m-1)}{m}\epsilon S\int_0^{x-\alpha}\frac{v^{m-2}\,dv}{(v^2+\epsilon S)^{\frac{1}{2}}}. \quad \text{(B 3)}$$

By using (B 1)–(B 3) in (4.10) and (4.11) we obtain

$$\begin{aligned} K_0(x,\epsilon) = {} & \tfrac{1}{2}\{[(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}(x-\alpha) - (x-\alpha)^2 - \epsilon S\log(x-\alpha+[(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}) \\ & + \tfrac{1}{2}\epsilon S\log(\epsilon S)\}, \end{aligned} \quad \text{(B 4)}$$

$$\begin{aligned} K_n(x,\epsilon) = {} & [(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}\left\{\sum_{j=1}^{n}\frac{(-\epsilon S)^j(x-\alpha)^{2(n-j)+1}}{2(n+1)}\prod_{m=1}^{j}\left(\frac{2n+3-2m}{2n+2-2m}\right) + \frac{(x-\alpha)^{2n+1}}{2(n+1)}\right\} \\ & - \sum_{j=0}^{n} a_j(\epsilon S)^j\frac{(x-\alpha)^{2(n-j+1)}}{2(n-j+1)} + (-\epsilon S)^{n+1}\prod_{m=0}^{n}\left(\frac{2n+1-2m}{2n+2-2m}\right) \\ & \times[\log(x-\alpha+[(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}) - \tfrac{1}{2}\log(\epsilon S)], \quad n \geqslant 1; \end{aligned} \quad \text{(B 5)}$$

$$P_0 = [(x-\alpha)^2+\epsilon S]^{\frac{1}{2}} - (\epsilon S)^{\frac{1}{2}} - (x-\alpha), \quad \text{(B 6)}$$

$$\begin{aligned} P_n = {} & [(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}\left\{\frac{(x-\alpha)^{2n}}{2n+1} + \sum_{j=1}^{n}\frac{(-\epsilon S)^j(x-\alpha)^{2(n-j)}}{(2n+1)}\prod_{m=1}^{j}\left(\frac{2n+2-2m}{2n+1-2m}\right)\right\} \\ & - \sum_{j=0}^{n} a_j(\epsilon S)^j\frac{(x-\alpha)^{2(n-j)+1}}{2(n-j)+1} + \frac{(\epsilon S)^{n+\frac{1}{2}}(-1)^{n+1}}{(2n+1)}\prod_{m=1}^{n}\left(\frac{2n+2-2m}{2n+1-2m}\right), \quad n \geqslant 1. \end{aligned} \quad \text{(B 7)}$$

By replacing $x-\alpha$ by $\beta-x$ in (B 4)–(B 7) we obtain $\tilde{K}_n$ and $\tilde{P}_n$.

To expand these results asymptotically in ϵ we must first consider the functions g, h and t_j defined by

$$g(x,\epsilon) = [(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}, \quad h(x,\epsilon) = \log(x-\alpha+[(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}, \quad t_j(x,\epsilon) = (x-\alpha)^j. \tag{B 8}$$

These functions have power series in ϵ given by

$$g(x,\epsilon) = \sum_{k=0}^{\infty} g_k(x)\,\epsilon^k, \quad g_k(x) = \frac{1}{k!}\frac{d^k}{d\epsilon^k}[(x-\alpha)^2+\epsilon S]^{\frac{1}{2}}\big|_{\epsilon=0}, \tag{B 9}$$

$$h(x,\epsilon) = \log 2x + \sum_{k=1}^{\infty} h_k(x)\,\epsilon^k, \quad h_k(x) = \frac{1}{k!}\frac{d^k}{d\epsilon^k}\log[x-\alpha+g(x,\epsilon)]\big|_{\epsilon=0}, \tag{B 10}$$

$$t_j(x,\epsilon) = \sum_{k=0}^{\infty} t_{jk}(x)\,\epsilon^k. \tag{B 11}$$

In (B 11) the functions $t_{jk}(x)$ are obtained by inserting the expansion for $\alpha(\epsilon)$ given by (2.6) into $(x-\alpha)^j$ and collecting terms in powers of ϵ.

We define the functions $\tilde{g}, \tilde{h}$ and $\tilde{t}_j$ by replacing $x-\alpha$ by $\beta-x$ in the definitions (B 8). Then these functions have power series expansions in ϵ with coefficients $\tilde{g}_k$, $\tilde{h}_k$ and $\tilde{t}_{jk}$ which are given by making the same replacement in the expressions for g_k, h_k and t_{jk}.

By using these power series in (B 4)–(B 7), taking account of the estimates (A 2) and (A 5), and using the corresponding results for $\tilde{K}_n$ and $\tilde{P}_n$ we obtain

$$\begin{aligned} K_0+\tilde{K}_0 = \frac{1}{2}\Bigg\{ & \sum_{\substack{k=0\\ i+k\geqslant 1}}^{\infty}\sum_{i=0}^{\infty} \epsilon^{i+k}[g_k t_{1i}+\tilde{g}_k \tilde{t}_{1i}] - \sum_{i=1}^{\infty}\epsilon^i[t_{2i}+\tilde{t}_{2i}] \\ & - S\epsilon\left[\log\left\{\frac{4x(1-x)}{S(x)}\right\} + \sum_{k=1}^{\infty}\epsilon^k[h_k+\tilde{h}_k] - \log\epsilon\right]\Bigg\}, \end{aligned} \tag{B 12}$$

$$\begin{aligned} K_n+\tilde{K}_n = & \sum_{\substack{k=0\\ i+j+k\geqslant n+1}}^{\infty}\sum_{i=0}^{\infty}\sum_{j=1}^{n} \frac{(-S)^j}{2(n+1)}\prod_{m=1}^{j}\left(\frac{2n+3-2m}{2n+2-2n}\right)\epsilon^{i+j+k}[g_k t_{2(n-j)+1,i}+\tilde{g}_k \tilde{t}_{2(n-j)+1,i}] \\ & + \sum_{\substack{k=0\\ i+k\geqslant n+1}}^{\infty}\sum_{i=0}^{\infty}\frac{\epsilon^{i+k}}{2(n+1)}[g_k t_{2n+1,i}+\tilde{g}_k \tilde{t}_{2n+1,i}] \\ & - \sum_{\substack{i=0\\ i+j\geqslant n+1}}^{\infty}\sum_{j=0}^{n}\frac{a_j S^j \epsilon^{i+j}}{2(n-j+1)}[t_{2(n-j+1),i}+\tilde{t}_{2(n-j+1),i}] \\ & + (-S)^{n+1}\prod_{m=0}^{n}\left(\frac{2n+1-2m}{2n+2-2m}\right)\left[\epsilon^{n+1}\log\left\{\frac{4x(1-x)}{S}\right\}\right. \\ & \left. + \sum_{k=1}^{\infty}\epsilon^{1+n+k}(h_k+\tilde{h}_k) - \epsilon^{n+1}\log\epsilon\right], \end{aligned} \tag{B 13}$$

$$P_0-\tilde{P}_0 = \sum_{k=1}^{\infty}\epsilon^k(g_k-\tilde{g}_k) + \sum_{k=1}^{\infty}(\alpha_k-\beta_k)\,\epsilon^k, \tag{B 14}$$

$$\begin{aligned} P_n-\tilde{P}_n = & \sum_{\substack{k=0\\ i+k\geqslant n+1}}^{\infty}\sum_{i=0}^{\infty}\frac{\epsilon^{i+k}}{2n+1}[g_k t_{2n,i}-\tilde{g}_k \tilde{t}_{2n,i}] \\ & + \sum_{\substack{k=0\\ i+j+k\geqslant n+1}}^{\infty}\sum_{i=0}^{\infty}\sum_{j=1}^{n}\frac{(-S)^j\epsilon^{i+j+k}}{(2n+1)}\prod_{m=1}^{j}\left(\frac{2n+2-2m}{2n+1-2m}\right)[g_k t_{2(n-j),i}-\tilde{g}_k \tilde{t}_{2(n-j),i}] \\ & - \sum_{\substack{i=0\\ i+j\geqslant n+1}}^{\infty}\sum_{j=0}^{n}\frac{\epsilon^{i+j}a_j S^j}{2(n-j)+1}[t_{2(n-j)+1,i}-\tilde{t}_{2(n-j)+1,i}], \quad n \geqslant 1. \end{aligned} \tag{B 15}$$

No terms which involve half integer powers of ϵ appear in (B 14) and (B 15) since they cancel when $P_n - \tilde{P}_n$ is formed.

To expand $H_n(x,\epsilon)$ and $\tilde{H}_n(x,\epsilon)$ defined by (4.9), we use its Taylor series and obtain

$$H_n(x,\epsilon) - \tilde{H}_n(x,\epsilon) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!}\frac{d^k}{d\epsilon^k}\left\{\int_0^{x-\alpha} v^{-2n}\left[F(x-v) - \sum_{j=0}^{2n-1}\frac{F^{(j)}(x)\,v^j}{j!}\right]dv \right.$$
$$\left. - \int_0^{\beta-x} v^{-2n}\left[F(x+v) - \sum_{j=0}^{2n-1}\frac{F^{(j)}(x)\,v^j}{j!}\right]dv\right\}. \quad \text{(B 16)}$$

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